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Discretely Uniform Approximation of Continuous Functions

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This paper studies discretely uniform approximation of continuous functions and the associated discrete limit spaces. In particular, those conditions are established which must be satisfied in order that discretely uniform convergence exist and be surjective and that the corresponding limit space be a metric discrete limit space. Further, discretely uniform convergence is characterized by discretely continuous convergence. These results are applied to spaces of continuous functions defined on subsets of $\mathbb{R}^n$ and of continuous functionals defined on subspaces of a reflexive Banach space. This theory is of interest in connection with Galerkin approximations, finite element methods and singular perturbations in Banach spaces.

Discretely uniform approximation of continuous functions plays an important role in approximation methods of numerical analysis. This paper studies the associated discrete limit spaces. Using the concepts of limit superior and limit inferior of sequences of sets G, G_ι, $\iota \in I$ (cf. [4, VII-Section 5]), we shall investigate the conditions which must be satisfied in order that discretely uniform convergence exist and be surjective and that the associated limit space be a metric discrete limit space. Of particular interest in applications is the characterization of discretely uniform convergence by discretely continuous convergence. The first application of our results deals with spaces of continuous functions defined on subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. In the case of compact metric spaces, the condition $\operatorname{Lim} G_\iota = G$ can be characterized by means of the Hausdorff distance d in the form $d(G, G_\iota) \to 0$ $(\iota \in I)$ (cf. Section 4). Finally, we apply the following theory to the approximation of subspaces in a reflexive Banach space. In this way, we are able to generalize results in [8] for Hilbert spaces to Banach spaces. Our results will then confirm that the spaces of continuous linear functionals defined on these subspaces constitute metric discrete approximations. The results are of great interest in the theories of Galerkin approximations, finite element methods and singular perturbations in Banach spaces. This paper has been written while the author was a guest of the Mathematical Institute of Aarhus University (cf. [10]).

1. Discretely Uniform Approximation

Let us assume that M is a normal Hausdorff space. Let $\mathbb{K}$ denote either the real or the complex number field. Given a sequence of nonvoid subsets $G, G_\iota, \iota \in I$, of M, then let $C(M)$ denote the Banach space with the supremum norm of bounded continuous $\mathbb{K}$-valued functions on M, and similarly let $C(G)$, $C(G_\iota)$ denote the Banach spaces of bounded continuous $\mathbb{K}$-valued functions on $G, G_\iota, \iota \in I$. For the sake of notational simplicity, we shall use the same symbol $\|\cdot\|$ to denote the norms of all these spaces; thus

$$\|u_\iota\| = \sup_{x \in G_\iota} |u_\iota(x)|, \qquad u_\iota \in C(G_\iota), \qquad \iota \in I.$$

The *discretely uniform approximation* "lim" is a binary relation between sequences $(u_\iota) \in \Pi_\iota C(G_\iota)$ and functions $u \in C(G)$, defined by

$$(u_\iota) \text{ "lim" } u \Leftrightarrow \exists \hat{u} \in C(M): \hat{u}_G = u, \quad \sup_{x \in G_\iota} |u_\iota(x) - \hat{u}(x)| \to 0 \quad (\iota \in I).$$

The relation "lim" has the basic property that for any continuous function $v \in C(M)$ the sequence of restrictions $v_{G_\iota} = v \mid G_\iota$, $\iota \in I$, and the restriction $v_G = v \mid G$ satisfy

$$\forall v \in C(M): (v_{G_\iota}) \text{ "lim" } v_G. \tag{1}$$

The relation "lim" is said to be *functional* if

$$(u_\iota) \text{ "lim" } u \quad \textit{and} \quad (u_\iota) \text{ "lim" } v \Rightarrow u = v,$$

for every $u, v \in C(G)$, and every sequence $(u_\iota) \in \Pi_\iota C(G_\iota)$. In this case, the *discretely uniform convergence* lim exists, and is

$$\lim u_\iota = u \Leftrightarrow (u_\iota) \text{ "lim" } u,$$

for all continuous functions $u \in C(G)$, $u_\iota \in C(G_\iota)$, $\iota \in I$. By virtue of (1), the discretely uniform convergence lim has the property

$$\forall v \in C(M): \lim v_{G_\iota} = v_G. \tag{2}$$

In particular, for $v = 0$ the sequence of trivial functions $0_\iota \in C(G_\iota)$, $0 \in C(G)$, $\iota \in I$, satisfies $\lim 0_\iota = 0$.

In the case that "lim" is functional, the triple $C(G), \Pi_\iota C(G_\iota)$, lim constitutes a *discrete limit space* (cf. [9, 11]). This space is said to be a *metric discrete limit space* if the following condition

$$\lim \| u_\iota - v_\iota \| = 0 \Leftrightarrow \lim u_\iota = \lim v_\iota \tag{M}$$

holds for every pair of sequences $u_\iota, v_\iota \in C(G_\iota)$, $\iota \in I$, such that (u_ι) or (v_ι) converges discretely uniformly. If, additionally, the mapping lim is surjective, the space $C(G), \Pi_\iota C(G_\iota)$, lim is said to be a *metric discrete approximation*. The triple $C(G), \Pi_\iota C(G_\iota)$, lim is said to be a metric discrete limit space with *discretely convergent metrics* if the condition (M) is valid and the relation

$$\lim u_\iota = u, \qquad \lim v_\iota = v \Rightarrow \lim \| u_\iota - v_\iota \| = \| u - v \|,$$

is true for every pair of discretely uniformly convergent sequences u_ι, $v_\iota \in C(G_\iota)$, $\iota \in I$.

2. Convergence of Sequences of Sets

In this section, we establish and characterize the requirements (G0), (G1), (G2) for the sequence G, G_ι, $\iota \in I$, which will be used in the study of the discretely uniform convergence lim and the associated discrete limit space $C(G), \Pi_\iota C(G_\iota)$, lim. An essential tool in the following proofs is the well-known lemma of Urysohn and the theorem of Tietze (cf. Alexandroff–Hopf [1, I–Section 6]; Kuratowski [5, Section 14]).

Let G, G_ι, $\iota \in I$, be an arbitrary sequence of nonvoid subsets of the normal Hausdorff space M. The (closed) *limit superior* (cf. Hausdorff [4, VII–Section 5] of the sequence (G_ι), denoted by $\operatorname{Lim\,sup} G_\iota$, is the set of all points x in M with the property: for every open neighborhood O of x there exists a subsequence I' of I such that $O \cap G_\iota \neq \varnothing$ for all $\iota \in I'$. The (closed) *limit inferior* of the sequence (G_ι), denoted by $\operatorname{Lim\,inf} G_\iota$, is the set of all points x in M with the property: for every neighborhood O of x there exists an index $\nu \in I$ such that $O \cap G_\iota \neq \varnothing$ for all $\iota \geqslant \nu$, $\iota \in I$. Finally, the sequence of sets (G_ι) is said to *converge* to the set G, the *limit* of (G_ι), if

$$\operatorname{Lim\,inf} G_\iota = \operatorname{Lim\,sup} G_\iota = G.$$

We write $\operatorname{Lim} G_\iota = G$ if and only if the sequence (G_ι) converges to G.

(3) *The condition*

$$G \subset \operatorname{Lim\,sup} G_\iota \tag{G0}$$

is equivalent to the statement

$$\forall v \in C(M): \| v_G \| \leqslant \limsup \| v_{G_\iota} \|. \tag{G0'}$$

Proof. (i) Let v be any function in $C(M)$ and let x be any point in G. Since v is continuous, there exists, for every $\epsilon > 0$, an open neighborhood O of x such that $|v(x) - v(x')| < \epsilon$ whenever $x' \in O$. In view of (G0), there exist a subsequence I' of I and points $x_\iota \in O \cap G_\iota$, $\iota \in I'$. Hence $|v(x)| \leqslant \|v_{G_\iota}\| + \epsilon$ for all $\iota \in I'$ and thus $|v(x)| \leqslant \limsup \|v_{G_\iota}\| + \epsilon$, $x \in G$. Consequently, for every $\epsilon > 0$, we have $\|v_G\| \leqslant \limsup \|v_{G_\iota}\| + \epsilon$ which entails (G0).

(ii) If (G0) is not true, there exists a point $z \in G$ which does not belong to $\operatorname{Lim\,sup} G_\iota$. Thus there exist an open neighborhood O of z and an index $\nu \in I$ such that $O \cap G_\iota = \varnothing$ for all $\iota \geqslant \nu$, $\iota \in I$. The lemma of Urysohn implies the existence of a real continuous function $w \in C(M)$ having the properties $w \mid \{z\} = 1$, $w \mid \complement O = 0$ and $0 \leqslant w \leqslant 1$. This function w satisfies $\|w_G\| = 1$ as well as $w_{G_\iota} = 0$ for all $\iota \geqslant \nu$ and hence $\limsup \|w_{G_\iota}\| = 0$, which contradicts (G0′).

For brevity, denote by (G1) the following statement concerning G, G_ι, $\iota \in N$.

(G1) *For every open neighborhood O of $\bar{G}$ there exists an index $\nu \in I$ with the property that $G_\iota \subset O$ for all $\iota \geqslant \nu$, $\iota \in I$.*

It is clear that this condition is trivially fulfilled when $G_\iota \subset \bar{G}$, $\iota \in I$. The above permits the following characterization.

(4) *The condition* (G1) *is equivalent to the statement*

$$\forall v \in C(M)\colon \limsup \|v_{G_\iota}\| \leqslant \|v_G\|. \tag{G1′}$$

Proof. (i) Let v be an arbitrary function in $C(M)$. For every $\epsilon > 0$ let

$$O_\epsilon = \{x \in M \mid |v(x)| < \|v_G\| + \epsilon\}.$$

Obviously O_ϵ is an open neighborhood of $\bar{G}$. Using condition (G1), there exists an index $\nu \in I$ such that $G_\iota \subset O_\epsilon$ and thus $|v(x)| < \|v_G\| + \epsilon$ whenever $\iota \geqslant \nu$ and $x \in G_\iota$. Hence $\|v_{G_\iota}\| \leqslant \|v_G\| + \epsilon$, $\iota \geqslant \nu$, and also $\limsup \|v_{G_\iota}\| \leqslant \|v_G\| + \epsilon$ for every $\epsilon > 0$, which proves (G1′).

(ii) If (G1) does not hold, there exist an open neighborhood O of $\bar{G}$ and a subsequence I' of I such that $G_\iota \cap \complement O \neq \varnothing$ for all $\iota \in I'$. Since $\bar{G} \cap \complement O = \varnothing$, from the lemma of Urysohn one obtains a real continuous function $w \in C(M)$ such that $w \mid \bar{G} = 0$, $w \mid \complement O = 1$ and $0 \leqslant w \leqslant 1$. In this case $\|w_G\| = 0$, but $\|w_{G_\iota}\| = 1$, $\iota \in I'$, so that $\limsup \|w_{G_\iota}\| = 1$. Hence (G1′) is not valid.

The following theorem establishes an important characterization of condition (G1) by the limit superior of the sequence (G_ι). Here we make use of the interesting representation (cf. Hausdorff [4, p. 237])

$$\operatorname{Lim\,sup} G_\iota = \bigcap_\kappa \overline{\bigcup_{\iota \geq \kappa} G_\iota}. \tag{5}$$

(6) *Condition* (G1) *implies that* $\operatorname{Lim\,sup} G_\iota \subset \bar{G}$. *If* M *is compact, these two conditions are equivalent,*

$$\text{(G1)} \Leftrightarrow \operatorname{Lim\,sup} G_\iota \subset \bar{G}.$$

Proof. (i) Assuming the first statement were not true, then there exists a point $x \in \operatorname{Lim\,sup} G_\iota \cap \complement \bar{G}$. Since M is a Hausdorff space, the set $\{x\}$ is closed. The space M is normal and $\{x\} \cap \bar{G} = \emptyset$ so that there exist disjoint open sets O_0, O_1 such that $\{x\} \subset O_0$, $\bar{G} \subset O_1$. Condition (G1) implies the existence of an index $\nu \in I$ such that $G_\iota \subset O_1 \subset \complement O_0$ whenever $\iota \geq \nu$. But, since $x \in \operatorname{Lim\,sup} G_\iota$, there exists a subsequence I' such that $O_0 \cap G_\iota \neq \emptyset$, $\iota \in I'$, which contradicts $G_\iota \subset \complement O_0$, $\iota \geq \nu$.

(ii) Let now M be compact, let $\operatorname{Lim\,sup} G_\iota \subset \bar{G}$ and let O be an arbitrary open neighborhood of $\bar{G}$. The limit superior has the representation (5),

$$\operatorname{Lim\,sup} G_\iota = \bigcap_\kappa S_\kappa, \qquad S_\kappa = \overline{\bigcup_{\iota \geq \kappa} G_\iota}, \qquad \kappa \in I.$$

Here $S_\iota \subset S_\kappa$ for $\iota \geq \kappa$, so (S_κ) is a decreasing sequence of closed sets and $\bigcap S_\kappa \subset O$. Since M is compact, there exists an index $\nu \in I$ with the property

$$G_\iota \subset \overline{\bigcup_{\iota \geq \nu} G_\iota} = S_\nu = \bigcap_{\kappa \leq \nu} S_\kappa \subset O, \qquad \iota \geq \nu.$$

Finally, we want to characterize a condition for the limit inferior of the sequence (G_ι).

(7) *The condition*

$$G \subset \operatorname{Lim\,inf} G_\iota, \tag{G2}$$

is equivalent to the statement

$$\forall v \in C(M): \| v_G \| \leq \liminf \| v_{G_\iota} \|. \tag{G2'}$$

Proof. (i) Let v be an arbitrary function in $C(M)$ and let x be an arbitrary point in G. Since v is continuous, there exists, for every $\epsilon > 0$, an open neighborhood O of x such that $| v(x) - v(x')| < \epsilon$ for all $x' \in O$. Using condition (G2), we obtain an index $\nu \in I$ and points $x_\iota \in O \cap G_\iota$ for all $\iota \geqslant \nu$, $\iota \in I$. Hence $|v(x)| \leqslant \| v_{G_\iota} \| + \epsilon$ for all $\iota \geqslant \nu$ and thus $|v(x)| \leqslant \liminf \| v_{G_\iota} \| + \epsilon$, $x \in G$. This implies, for every $\epsilon > 0$, the relation $\| v_G \| \leqslant \liminf \| v_{G_\iota} \| + \epsilon$ and thus (G2′).

(ii) If (G2) does not hold, then there exists a point $z \in G \cap \complement \operatorname{Lim\,inf} G_\iota$. Hence there exist an open neighborhood O of x and a subsequence I' of I such that $O \cap G_\iota = \varnothing$ or $G_\iota \subset \complement\, O$ for all $\iota \in I'$. From the lemma of Urysohn it follows that there exists a real continuous function $w \in C(M)$ with the properties $w \mid \{z\} = 1$, $w \mid \complement\, O = 0$ and $0 \leqslant w \leqslant 1$. Using this function w, we have $\| w_G \| = 1$ and $\| w_{G_\iota} \| = 0$, $\iota \in I'$, so that (G2′) is not true.

3. Discretely Uniform Limit Spaces

We are now in a position to establish the fundamental theorems concerning discretely uniform convergence. As in the preceding section, let G, G_ι, $\iota \in I$, be an arbitrary sequence of nonvoid subsets of the normal Hausdorff space M.

(8) *If G is closed, there exists for every continuous function $u \in C(G)$ an extension $\hat{u} \in C(M)$ with the property*

$$\hat{u}_G = u, \qquad (\hat{u}_{G_\iota}) \text{ “lim” } u, \tag{9}$$

so that the discretely uniform approximation “lim” *is surjective.*

Proof. If G is closed, for every real-valued function $u \in C(G)$ the extension theorem of Tietze implies the existence of a continuous extension $\hat{u} \in C(M)$ such that $\hat{u}_G = u$. If $u \in C(G)$ is complex-valued, there exist extensions $\hat{u}_{\mathrm{Re}}$, $\hat{u}_{\mathrm{Im}}$ of the real and imaginary parts u_{Re}, u_{Im} of u, defined by $u(x) = u_{\mathrm{Re}}(x) + iu_{\mathrm{Im}}(x)$, $x \in G$. On setting $\hat{u} = \hat{u}_{\mathrm{Re}} + i\hat{u}_{\mathrm{Im}}$ we obtain an extension with the property $\hat{u}_G = u$. It is clear that in either case the relation $(\hat{u}_{G_\iota})$ “lim” u holds.

(10) *The discretely uniform convergence* lim *exists and* $C(G)$, $\Pi_\iota C(G_\iota)$, lim *is a discrete limit space if and only if the condition* (G0) *holds. Under the additional assumption that G is closed, the discretely uniform convergence* lim *is surjective.*

Proof. We first prove that under the assumption (G0) the relation “lim” is functional. For every (u_ι), u and (u_ι), v which satisfy the relation “lim”, there are functions $\hat{u}, \hat{v} \in C(M)$ with the property $\hat{u} \mid G = u$, $\hat{v} \mid G = v$ and

$$\lim \| u_\iota - \hat{u}_{G_\iota} \| = 0, \qquad \lim \| u_\iota - \hat{v}_{G_\iota} \| = 0.$$

Hence the function $w = \hat{u} - \hat{v} \in C(M)$ satisfies

$$\| w_{G_\iota} \| = \| \hat{u}_{G_\iota} - \hat{v}_{G_\iota} \| \leqslant \| u_\iota - \hat{u}_{G_\iota} \| + \| u_\iota - \hat{v}_{G_\iota} \| \to 0 \qquad (\iota \in I).$$

The condition (G0) implies (G0′) so that $w_G = 0$ or $u - v = \hat{u}_G - \hat{v}_G = 0$. Finally, if G is closed, we obtain from theorem (8) that lim is surjective.

(ii) Conversely, (G0) is a necessary condition. For if (G0) is not true, there exist a point $z \in G$, an open neighborhood O of z and an index $\nu \in I$ such that $O \cap G_\iota = \emptyset$ for all $\iota \geqslant \nu$, $\iota \in I$. The lemma of Urysohn assures the existence of a real continuous function w with the properties $w \mid \{z\} = 1$, $w \mid \complement O = 0$ and $0 \leqslant w \leqslant 1$. Hence $w_{G_\iota} = 0$ for all $\iota \geqslant \nu$ and

$$\lim \| w_{G_\iota} - 0_{G_\iota} \| = 0, \qquad \lim \| w_{G_\iota} - w_{G_\iota} \| = 0.$$

Consequently, (w_{G_ι}), 0_G as well as (w_{G_ι}), w_G satisfy the relation "lim", but $0_G \neq w_G$ so that "lim" cannot be functional.

We now come to the main result of this section.

(11) *The discretely uniform convergence* lim *exists and* $C(G)$, $\Pi_\iota C(G_\iota)$, lim *constitutes a metric discrete limit space if and only if the conditions* (G0), (G1) *are valid. Under these conditions, the discretely uniform convergence is equivalent to the following relation*

$$\lim u_\iota = u \Leftrightarrow \sup_{x \in G_\iota} | u_\iota(x) - \hat{u}(x)| \to 0 \qquad (\iota \in I), \tag{12}$$

for every sequence of continuous functions $u \in C(G)$, $u_\iota \in C(G_\iota)$, $\iota \in I$, *and for an arbitrary extension* $\hat{u} \in C(M)$ *of* u *such that* $\hat{u} \mid G = u$.

Proof. (i) Suppose (G0), (G1) be valid. From theorem (10) we obtain that the discretely uniform convergence lim exists and that $C(G)$, $\Pi_\iota C(G_\iota)$, lim is a discrete limit space. Let (u_ι), $(v_\iota) \in \Pi_\iota C(G_\iota)$ be any pair of sequences such that (u_ι) or (v_ι) is discretely uniformly convergent. Assume, for instance, (u_ι) converges to u. In this case, there exists a function $\hat{u} \in C(M)$ with the property $\hat{u}_G = u$ and $\lim \| u_\iota - \hat{u}_{G_\iota} \| = 0$. If $\lim \| u_\iota - v_\iota \| = 0$, then we have

$$\| v_\iota - \hat{u}_{G_\iota} \| \leqslant \| u_\iota - v_\iota \| + \| u_\iota - \hat{u}_{G_\iota} \| \to 0 \qquad (\iota \in I),$$

so that (v_ι) is discretely uniformly convergent and $\lim u_\iota = \lim v_\iota$. Conversely, if (u_ι) and (v_ι) are discretely uniformly convergent to $\lim u_\iota = \lim v_\iota$, there exist functions $\hat{u}, \hat{v} \in C(M)$ such that $\hat{u}_G = \hat{v}_G = u$ and

$$\lim \| u_\iota - \hat{u}_{G_\iota} \| = 0, \qquad \lim \| v_\iota - \hat{v}_{G_\iota} \| = 0.$$

Hence $w = \hat{u} - \hat{v} \in C(M)$ and $w_G = 0$. By Theorem (4), the condition (G1′) is valid. Consequently, it follows that $\lim \| w_{G_\iota} \| = 0$. Hence

$$\| u_\iota - v_\iota \| \leqslant \| u_\iota - \hat{u}_{G_\iota} \| + \| w_{G_\iota} \| + \| v_\iota - \hat{v}_{G_\iota} \| \to 0 \qquad (\iota \in I).$$

This shows that the condition (M) is valid and so $C(G)$, $\Pi_\iota C(G_\iota)$, lim is a metric discrete limit space. Finally, let (u_ι) be any discretely uniformly convergent sequence and let $\lim u_\iota = u$. Then there exists a function $\hat{v} \in C(M)$ such that $\hat{v}_G = u$ and $\lim \| u_\iota - \hat{v}_{G_\iota} \| = 0$. Let $\hat{u} \in C(M)$ be an arbitrary extension of $u = \hat{u}_G$. On setting $w = \hat{u} - \hat{v}$ and using condition (G1′), it follows that

$$| \| u_\iota - \hat{u}_{G_\iota} \| - \| u_\iota - \hat{v}_{G_\iota} \| | \leqslant \| w_{G_\iota} \| \to 0 \qquad (\iota \in I),$$

which entails (12).

(ii) Conversely, if "lim" is functional, Theorem (10) ensures the validity of condition (G0). For every $v \in C(M)$, the sequence (v_{G_ι}) is discretely uniformly convergent to $\lim v_{G_\iota} = v_G$. In particular, we have $\lim 0_\iota = 0$. If $C(G)$, $\Pi_\iota C(G_\iota)$, lim is a metric discrete limit space, we have, for every $v \in C(M)$ such that $v_G = 0$, the equivalence

$$\lim v_{G_\iota} = 0 \Leftrightarrow \lim \| v_{G_\iota} \| = 0.$$

For every open neighborhood O of $\bar{G}$, the lemma of Urysohn affirms the existence of a real continuous function w such that $w \mid \bar{G} = 0$, $w \mid \complement O = 1$ and $0 \leqslant w \leqslant 1$. On setting $v = w$ in the above equivalence, we see that $\lim \| w_{G_\iota} \| = 0$. Hence there exists an index $\nu \in I$ such that $| w(x) | \leqslant \| w_{G_\iota} \| < 1$ for all $x \in G_\iota$ and all $\iota > \nu$, $\iota \in I$. Consequently, $G_\iota \subset O$ for every $\iota > \nu$ which proves condition (G1).

By virtue of Theorems (3) and (4), the assumption in the above main theorem permits the characterization

$$\text{(G0) and (G1)} \Leftrightarrow \forall v \in C(M)\text{: } \lim\sup \| v_{G_\iota} \| = \| v_G \|. \qquad (13)$$

When M is compact, one has the equivalence

$$\text{(G0) and (G1) and } G \text{ closed} \Leftrightarrow \operatorname{Lim\,sup} G_\iota = G. \qquad (14)$$

Since every compact Hausdorff space is normal, we can state the following corollary of Theorem (11).

(15) *Let M be a compact Hausdorff space and let G be closed. Then the discretely uniform convergence* lim *exists and* $C(G)$, $\Pi_\iota C(G_\iota)$, lim *constitutes a metric discrete approximation if and only if* $\operatorname{Lim\,sup} G_\iota = G$.

The condition (G2) implies the condition (G0). Using Theorems (4) and (7), the following equivalence holds,

$$\text{(G1) and (G2)} \Leftrightarrow \forall v \in C(M): \lim \|v_{G_\iota}\| = \|v_G\|. \tag{16}$$

When M is compact, we can further state the interesting characterization

$$\text{(G1) and (G2) and } G \text{ closed} \Leftrightarrow \operatorname{Lim} G_\iota = G. \tag{17}$$

As one easily sees, this leads to the following theorem.

(18) *Let M be a compact Hausdorff space and let G be closed. Then the discretely uniform convergence* lim *exists and* $C(G), \Pi_\iota C(G_\iota)$, lim *is a metric discrete limit space with discretely convergent metrics if and only if* $\operatorname{Lim} G_\iota = G$.

Of particular interest in the applications of this theory is the relationship between discretely uniform convergence and discretely continuous convergence of sequences of continuous functions. Every sequence of nonvoid subsets $G, G_\iota \subset M$, $\iota \in I$, is formed into a discrete limit space $G, \Pi_\iota G_\iota$, $\lim^M$ by the convergence $\lim^M$ of the topological space M. Every sequence of $\mathbb{K}$-valued functions $u \in C(G)$, $u_\iota \in C(G_\iota)$ may be viewed as a sequence of mappings $u: G \to \mathbb{K}$, $u_\iota: G_\iota \to \mathbb{K}$, $\iota \in I$. If $\lim^M(\Pi_\iota G_\iota, G)$ is surjective, the discrete convergence $u_\iota \to u$ $(\iota \in I)$ is defined by the relation

$$\lim^M x_\iota = x \Rightarrow \lim u_\iota(x_\iota) = u(x),$$

for every sequence of points $x \in G$, $x_\iota \in G_\iota$, $\iota \in I$. Let us call this convergence the *discretely continuous convergence*. Then the following remarkable statement holds.

(19) *Let* $\lim^M(\Pi_\iota G_\iota, G)$ *be surjective. Then the discretely uniform convergence of* (u_ι) *to* u *implies the discretely continuous convergence of* (u_ι) *to* u. *If, in addition, M is sequentially compact and* $\operatorname{Lim} G_\iota = G$, *then the two convergences are equivalent.*

Proof. (i) If $\lim^M(\Pi_\iota G_\iota, G)$ is surjective, one has the relation $G \subset \operatorname{Lim\,inf} G_\iota \subset \operatorname{Lim\,sup} G_\iota$ such that the conditions (G0) and (G2) are valid. Hence the discretely uniform convergence exists. Let (u_ι) be discretely uniformly convergent to u. Then there exists an extension $\hat{u} \in C(M)$ of $u = \hat{u}_G$ such that $\lim \|u_\iota - \hat{u}_{G_\iota}\| = 0$. For every convergent sequence of points $x \in G$, $x_\iota \in G_\iota$ such that $\lim^M x_\iota = x$ $(\iota \in I)$ it follows that $\lim \hat{u}(x_\iota) = u(x)$, because $\hat{u}$ is continuous. Consequently,

$$|u_\iota(x_\iota) - u(x)| \leqslant |\hat{u}(x_\iota) - u(x)| + \|u_\iota - \hat{u}_{G_\iota}\| \to 0 \qquad (\iota \in I).$$

(ii) Suppose, in addition, that M be sequentially compact and $\operatorname{Lim} G_\iota = G$. Then $u_\iota \to u$ ($\iota \in I$) implies $\| u_\iota - \hat{u}_{G_\iota} \| \to 0$ ($\iota \in I$). Assume this were not true. Then, for every extension $\hat{u} \in C(M)$ of $u = \hat{u}_G$, there exist an $\epsilon_0 > 0$, a subsequence I' of I and points $z_\iota \in G_\iota$, $\iota \in I$, with the property

$$| u_\iota(z_\iota) - \hat{u}(z_\iota)| > \epsilon_0 , \qquad \iota \in I'.$$

Since M is sequentially compact, there further exist a subsequence I'' of I', and a point $z \in M$ such that $z_\iota \to z$ ($\iota \in I''$) and hence $z \in \operatorname{Lim\,sup} G_\iota \subset G$. By assumption, $\lim^M(\Pi_\iota G_\iota , G)$ is surjective so that one has a sequence of points $z_\iota' \in G_\iota$, $\iota \in I$, converging to z. On setting $z_\iota = z_\iota'$, $\iota \in I - I''$, we obtain an extended sequence (z_ι) converging to z as well. The continuity of $\hat{u}$ implies $\hat{u}(z_\iota) \to u(z)$ ($\iota \in I$) and the discretely continuous convergence of (u_ι) to u implies $u_\iota(z_\iota) \to u(z)$ ($\iota \in I$) which contradicts the above inequality.

4. Applications

The aim of this section is to illustrate the scope of our results by typical examples.

Example 1. (i) Let $M = [a_0 , b_0]$ be a compact interval of the real line and let $I = (1, 2, 3,\ldots)$ be the sequence of natural numbers. Let $[a, b]$ be a closed subinterval of $[a_0 , b_0]$ and let $(t_1 , t_2 , t_3 ,\ldots)$ be an infinite sequence of numbers in $[a_0 , b_0]$. Let

$$G = [a, b], \qquad G_\iota = \{t_\iota\}, \qquad \iota = 1, 2, \ldots .$$

Then one easily shows that the following equivalences hold,

$$(\text{G0}) \Leftrightarrow [a, b] \subset \operatorname{Lim\,sup}\{t_\iota\}$$
$$\Leftrightarrow \forall t \in [a, b]\ \exists I' \leqslant I\colon \lim_{\iota \in I'} t_\iota = t$$

and

$$(\text{G1}) \Leftrightarrow \operatorname{Lim\,sup}\{t_\iota\} \subset [a, b]$$
$$\Leftrightarrow a \leqslant \liminf t_\iota , \quad \limsup t_\iota \leqslant b.$$

However, condition (G2) or $[a, b] \subset \operatorname{Lim\,inf}\{t_\iota\}$ cannot be satisfied when $a < b$. If (G0), (G1) are true, the discretely uniform convergence has the form

$$\lim u_\iota = u \Leftrightarrow | u_\iota(t_\iota) - \hat{u}(t_\iota)| \to 0 \qquad (\iota \to \infty),$$

for all sequences of numbers $u_\iota(t_\iota)$, $\iota = 1, 2, \ldots$, all functions $u \in C[a, b]$ and any extension $\hat{u} \in C[a_0, b_0]$ such that $\hat{u} \mid [a, b] = u$.

(ii) Let $M = [a_0, b_0]$ and let $G = [a, b]$ as above. Now suppose (G_ι) be specified by a sequence of finite partitions,

$$G_\iota = \{t_0^\iota, \ldots, t_{N_\iota}^\iota\}, \qquad a_0 \leqslant a_\iota = t_0^\iota < t_1^\iota < \cdots < t_{N_\iota}^\iota = b_\iota \leqslant b_0,$$

for $\iota = 1, 2, \ldots$ such that

$$h_\iota = \max_{k=1,\ldots,N_\iota} (t_k^\iota - t_{k-1}^\iota) \to 0 \qquad (\iota \to \infty).$$

Then

$$\begin{aligned}(\mathrm{G0}) &\Leftrightarrow [a, b] \subset \operatorname{Lim\,sup}\{t_0^\iota, \ldots, t_{N_\iota}^\iota\} \\ &\Leftrightarrow \liminf a_\iota \leqslant a, \quad b \leqslant \limsup b_\iota\end{aligned}$$

and

$$\begin{aligned}(\mathrm{G1}) &\Leftrightarrow \operatorname{Lim\,sup}\{t_0^\iota, \ldots, t_{N_\iota}^\iota\} \subset [a, b] \\ &\Leftrightarrow a \leqslant \liminf a_\iota, \quad \limsup b_\iota \leqslant b\end{aligned}$$

so that

$$\begin{aligned}(\mathrm{G0}) \text{ and } (\mathrm{G1}) &\Leftrightarrow \operatorname{Lim\,sup}\{t_0^\iota, \ldots, t_{N_\iota}^\iota\} = [a, b] \\ &\Leftrightarrow \liminf a_\iota = a, \quad \limsup b_\iota = b.\end{aligned}$$

Finally.

$$\begin{aligned}(\mathrm{G1}) \text{ and } (\mathrm{G2}) &\Leftrightarrow \operatorname{Lim}\{t_0^\iota, \ldots, t_{N_\iota}^\iota\} = [a, b] \\ &\Leftrightarrow \lim a_\iota = a, \quad \lim b_\iota = b.\end{aligned}$$

EXAMPLE 2. (Cf. Carathéodory [2, pp. 172–182]). Let M be a compact n-dimensional interval of the euclidean space $\mathbb{R}^n$. Let G, G_ι, $\iota \in I$, be a sequence of subsets of M such that $\operatorname{Lim} G_\iota = G$ and let $u \in C(G)$, $u_\iota \in C(G_\iota)$, $\iota \in I$, be a sequence of bounded continuous functions. Then Theorem (19) affirms the equivalence of the discretely uniform convergence

$$\sup_{x \in G_\iota} | u_\iota(x) - \hat{u}(x) | \to 0 \qquad (\iota \in I),$$

where $\hat{u}$ denotes an arbitrary continuous extension of u, and of the discretely continuous convergence

$$\lim{}^{\mathbb{R}^n} x_\iota = x \Rightarrow \lim u_\iota(x_\iota) = u(x),$$

for all sequences of points $x \in G$, $x_\iota \in G_\iota$, $\iota \in I$.

EXAMPLE 3. Let M be a metric space with distance $|\cdot,\cdot|$ and let G, G_ι, $\iota \in I$, be a sequence of nonvoid subsets of M. Then the limit superior and the limit inferior of (G_ι) can be characterized by means of distances (cf. Kuratowski [5, Section 29]). In this way, we obtain

$$\begin{aligned}(\mathrm{G0}) &\Leftrightarrow G \subset \operatorname{Lim\,sup} G_\iota \\ &\Leftrightarrow \forall x \in G\colon \liminf |x, G_\iota| = 0\end{aligned}$$

and

$$\begin{aligned}(\mathrm{G2}) &\Leftrightarrow G \subset \operatorname{Lim\,inf} G_\iota \\ &\Leftrightarrow \forall x \in G\colon \limsup |x, G_\iota| = 0.\end{aligned}$$

Let us now assume M be compact and G be closed. Then one easily shows that our conditions (G1), (G2) assume the form

$$\begin{aligned}(\mathrm{G1}) &\Leftrightarrow \operatorname{Lim\,sup} G_\iota \subset G \\ &\Leftrightarrow \lim(\sup_{x_\iota \in G_\iota} |x_\iota, G|) = 0\end{aligned}$$

and

$$\begin{aligned}(\mathrm{G2}) &\Leftrightarrow G \subset \operatorname{Lim\,inf} G_\iota \\ &\Leftrightarrow \lim(\sup_{x \in G} |x, G_\iota|) = 0.\end{aligned}$$

Finally, introduce the Hausdorff distance between sets (cf. Hausdorff [4, p. 293]), defined by

$$d(A, B) = \max(\sup_{x \in A} |x, B|, \sup_{y \in B} |y, A|)$$

for subsets $A, B \subset M$. Then we obtain the characterization

$$\operatorname{Lim} G_\iota = G \Leftrightarrow \lim d(G, G_\iota) = 0.$$

EXAMPLE 4. Our results may be applied to the discretely uniform convergence of sequences of bounded linear functionals on continuously embedded subspaces of a reflexive Banach space. In this way, we obtain very interesting conditions which affirm that the dual spaces constitute metric discrete approximations. These results extend those of Stummel [8] for the case of Hilbert spaces.

Let E be a reflexive Banach space with norm $\|\cdot\|_E$. Suppose given a sequence F, F_ι, $\iota \in I$, of continuously embedded subspaces of E. That is $F, F_\iota \subset E$, $\iota \in I$, as vector spaces and there is a sequence of norms $\|\cdot\|_F$, $\|\cdot\|_{F_\iota}$ such that

$$\|x\|_E = \|x\|_F, \quad x \in F, \qquad \|x_\iota\|_E \leqslant \rho \|x_\iota\|_{F_\iota}, \quad x_\iota \in F_\iota, \quad \iota \in I, \quad (20)$$

with some number $\rho \geq 1$. The closed ball $B_\rho^E(0)$ of radius ρ and center 0 in E is a compact topological space in the weak topology, that is the E'-topology of E. Further $B_\rho^E(0)$ is weakly sequentially compact (cf. Dunford–Schwartz [3, Chapter V]). A sequence $(u_\iota) \in \Pi_\iota B_\rho^E(0)$ converges to u in the weak topology, if and only if

$$w\text{-lim } u_\iota = u \Leftrightarrow \forall 1 \in E' \colon 1(u_\iota) \to 1(u) \qquad (\iota \in I).$$

Further, let us denote by $B_1^F(0)$, $B_1^{F_\iota}(0)$, the closed unit balls in F resp. F_ι for all $\iota \in I$. By (20), we have that $B_1^F(0)$, $B_1^{F_\iota}(0) \subset B_\rho^E(0)$ and that $B_1^F(0)$ is closed. So let

$$M = B_\rho^E(0), \qquad G = B_1^F(0), \qquad G_\iota = B_1^{F_\iota}(0), \qquad \iota \in I.$$

Given a sequence of continuous linear functionals $v \in F'$, $v_\iota \in F_\iota'$, $\iota \in I$, we can obviously view these functionals as continuous functions $v \in C(B_1^F(0))$, $v_\iota \in C(B_1^{F_\iota}(0))$, $\iota \in I$. Under the assumption

$$w\text{-Lim sup } B_1^{F_\iota}(0) = B_1^F(0),$$

we obtain from Corollary (15) that the discretely uniform convergence lim exists and that this convergence is defined by

$$\lim v_\iota = v \Leftrightarrow \sup_{\|x_\iota\|_{F_\iota} \leq 1} | v_\iota(x_\iota) - \hat{v}(x_\iota)| \to 0 \qquad (\iota \in I)$$

for any extension $\hat{v}$ of v such that $\hat{v} \mid F = v$ and, due to the theorem of Hahn–Banach, $\hat{v} \in E'$. Obviously

$$\sup_{\|x_\iota\|_{F_\iota} \leq 1} | v_\iota(x_\iota) - \hat{v}(x_\iota)| = \sup_{0 \neq u_\iota \in F_\iota} | v_\iota(u_\iota) - \hat{v}(u_\iota)|/\| u_\iota \|_{F_\iota}$$

$$= \| v_\iota - \hat{v}_{F_\iota} \|_{F_\iota'}, \quad \hat{v}_{F_\iota} = \hat{v} \mid F_\iota, \quad \iota \in I,$$

so that

$$\lim v_\iota = v \Leftrightarrow \lim \| v_\iota - \hat{v}_{F_\iota} \|_{F_\iota'} = 0.$$

Moreover, under the assumption

$$w\text{-Lim } B_1^{F_\iota}(0) = B_1^F(0),$$

we have that

$$\lim v_\iota = v \Rightarrow \lim \| v_\iota \|_{F_\iota'} = \| v \|_{F'} .$$

Finally, let us apply Theorem (19). Suppose, additionally, that

$$w\text{-lim}(\Pi_\iota B_1^{F_\iota}(0), B_1^{F}(0)) \text{ is surjective.}$$

Then the discretely uniform convergence $\lim \| v_\iota - \hat{v}_{F_\iota} \|_{F_\iota'} = 0$ is equivalent to the discretely continuous convergence,

$$w\text{-lim } u_\iota = u \Rightarrow \lim v_\iota(u_\iota) = v(u),$$

for every sequence of elements $u \in B_1^F(0)$, $u_\iota \in B_1^{F_\iota}(0)$, $\iota \in I$.

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